W2L6 - HOMOGENEOUS LINEAR ODES
(with constant coefficients)

Consider the equation $ay'' + by' + cy = 0$ ← Eq. 2
Recall that if $y' = my$ then $e = e^{mt}$ is a solution.
We hypothesize that a solution might be of the same form: $y = e^{mt}$

Question: For what values of m is $y = e^{mt}$ a solution for Eq. 2?

Plug it in: $y' = me^{mt}$ and $y'' = m^2 e^{mt}$

$$ay'' + by' + cy = 0$$
$$a(m^2 e^{mt}) + b(me^{mt}) + c(e^{mt}) = 0$$
$$\underbrace{e^{mt}}_{\text{never zero}}(am^2 + bm + c) = 0$$
$$\Leftrightarrow \quad am^2 + bm + c \leftarrow \text{characteristic polynomial}$$
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 CASES:

1. 2 distinct real roots (m_1, m_2)
$\underbrace{y_1 = e^{m_1 t} \qquad y_2 = e^{m_2 t}}_{\text{linearly independent when } m_1 \neq m_2}$ $\Rightarrow \boxed{y = C_1 e^{m_1 t} + C_2 e^{m_2 t}}$

2. 1 root (repeated) $m_1 = m_2$
$\underbrace{y_1 = e^{m_1 t} \qquad y_2 = te^{m_1 t}}_{\text{linearly independent}}$ $\Rightarrow \boxed{y = C_1 e^{m_1 t} + C_2 t e^{m_1 t}}$

3. Complex Conjugates $(a \pm bi)$ $\quad m_1 = a + bi \quad m_2 = a - bi$
$y_1 = e^{(a+bi)t} \qquad y_2 = e^{(a-bi)t}$
$$\Rightarrow y = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} = \boxed{e^{at}(C_1 e^{bit} + C_2 e^{-bit})}$$

Recall: Euler's Formula
$$e^{\theta i} = \cos\theta + i\sin\theta$$

Question: Can we re-write the complex form in Case 3 without any i's?

$$e^{bit} = e^{(bt)i} = \cos bt + i \sin bt$$
$$e^{-bit} = e^{(-bt)i} = \cos(-bt) + i \sin(-bt)$$
$$= \cos(bt) - i \sin(bt)$$

$\frac{1}{2}e^{bit} + \frac{1}{2}e^{-bit} = \cos bt$
and $\frac{1}{2}e^{bit} - \frac{1}{2}e^{-bit} = i\sin bt$

Recall: $y = e^{at}(C_1 e^{bit} + C_2 e^{-bit})$ was a solution for all C_1, C_2

If $C_1 = C_2 = \frac{1}{2}$: $y = e^{at}\left(\frac{1}{2}e^{bit} + \frac{1}{2}e^{-bit}\right) = e^{at}\cos bt$ ← a solution

If $C_1 = \frac{1}{2}, C_2 = \frac{-1}{2}$: $y = e^{at}\left(\frac{1}{2}e^{bit} - \frac{1}{2}e^{-bit}\right) = e^{at}\, i \sin bt$ $\Rightarrow y = e^{at}\sin bt$ — also a solution

* General Solution: $\underline{y = C_1 e^{at}\cos bt + C_2 e^{at}\sin bt}$

EX: Find the general solution for the equation: $2y'' - 5y' - 3y = 0$

Characteristic Polynomial: $2m^2 - 5m - 3 = 0 \Rightarrow (2m+1)(m-3)$

$m = -\frac{1}{2}, 3$

$\Rightarrow \underline{y = C_1 e^{-\frac{1}{2}t} + C_2 e^{3t}}$

EX: Find the general solution for the equation: $y'' - 10y' + 25y = 0$

$m^2 - 10m + 25 = 0 \Rightarrow (m-5)^2 = 0 \Rightarrow m = 5, 5$

$\Rightarrow \underline{y = C_1 e^{5t} + C_2 t e^{5t}}$

EX: Find the general solution for the equation: $y'' + 4y' + 7y = 0$

$m^2 + 4m + 7 = 0 \qquad m = \frac{-4 \pm \sqrt{16 - 4(7)}}{2} = -2 \pm i\sqrt{3}$

$y = C_1 e^{-2t}\cos\sqrt{3}t + C_2 e^{-2t}\sin\sqrt{3}t$

or

$y = e^{-2t}(C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t)$

EX: Solve the IVP: $\begin{cases} 4y'' + 4y' + 17y = 0 \\ y(0) = -1; \; y'(0) = 2 \end{cases}$

$4m^2 + 4m + 17 = 0 \Rightarrow m = \frac{-4 \pm \sqrt{16 - 4(4)(17)}}{8} \Rightarrow \frac{-4 \pm \sqrt{-256}}{8} \Rightarrow \frac{-4 \pm 16i}{8}$

$m = -\frac{1}{2} \pm 2i$

$y = C_1 e^{-\frac{1}{2}t}\cos 2t + C_2 e^{-\frac{1}{2}t}\sin 2t$

$y(0) = C_1 \underbrace{e^{-\frac{1}{2}(0)}}_{1} \underbrace{\cos 2(0)}_{1} + C_2 \underbrace{e^{-\frac{1}{2}(0)}}_{1} \underbrace{\sin 2(0)}_{0} = -1$

$C_1 = -1$

$y' = e^{-\frac{1}{2}t}(C_1\cos 2t + C_2\sin 2t)$

$\Rightarrow y' = -\frac{1}{2}e^{-\frac{1}{2}t}(C_1\cos 2t + C_2\sin 2t) + e^{-\frac{1}{2}t}(-2C_1\sin 2t + 2C_2\cos 2t)$

$y'(0) = -\frac{1}{2}C_1 + 2C_2 = 2$

$-\frac{1}{2}(-1) + 2C_2 = 2$

$2C_2 = \frac{3}{2} \qquad C_2 = \frac{3}{4}$

$\underline{y = -e^{-\frac{1}{2}t}\cos 2t + \frac{3}{4}e^{-\frac{1}{2}t}\sin 2t}$

HIGHER ORDER EQNS
(nth order, homogeneous with constant coefficients)

Consider the equation:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \qquad \text{(Eq. 4)}$$

The characteristic polynomial for Equ. 4 is given by

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

To find a general solution we solve the characteristic polynomial for m:

Cases:

1. Distinct Roots: $y_i = e^{m_i x}$
2. Repeated Roots: If m_i is repeated k times then use $e^{m_i x}, xe^{m_i x}, x^2 e^{m_i x}, \ldots, x^{k-1} e^{m_i x}$
3. Complex Conjugates: If $\alpha \pm \beta i$ are roots, then use $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$

EX: Find the general solution for $\underbrace{y''' + 3y'' - 4y}_{m^3 + 3m^2 - 4 = 0} = 0$

$$(m-1)(m+2)^2 \Rightarrow m = 1, 2, 2$$

$$y = C_1 e^{t} + C_2 e^{-2t} + C_3 t e^{-2t}$$

EX: Assume that the characteristic polynomial for Eq. 4 has the following roots:

$$m = -1, -1, -1, \quad 2, \quad -1+i, \ -1-i$$

What is the general solution to the ODE?

$$y = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t} + C_4 e^{2t} + C_5 e^{-t}\cos t + C_6 e^{-t}\sin t$$